

Primordial black hole formation and hybrid inflation

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We calculate the contribution to the curvature perturbation, that is generated while the waterfall field grows exponentially (which can occur only if the waterfall field mass is $m \gg H$). We consider the upper bound on its spectrum coming from constraints on the abundance of primordial black holes, taking account of possible non-gaussianity. The constraint is satisfied, but extrapolation of our result to the regime $m \sim H$ suggests that the constraint might not be satisfied there.

I. INTRODUCTION

On cosmological scales, the primordial curvature perturbation ζ is very small with spectrum $\mathcal{P}_\zeta(k) \simeq (5 \times 10^{-5})^2$. But $\mathcal{P}_\zeta(k)$ may rise as the wavenumber k increases, to some much bigger peak value $\mathcal{P}_\zeta(k_{\text{peak}})$. If that happens, primordial black holes may form as k_{peak} enters the horizon, and cosmological bounds on their abundance translate to an upper bound on $\mathcal{P}_\zeta(k_{\text{peak}})$.

The bound is usually discussed under the assumption that ζ is nearly gaussian, giving very roughly $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 10^{-2}$. In the first part of this paper we discuss the bound assuming instead that on scales around k_{peak}

$$\zeta(\mathbf{x}) = \pm (g^2(\mathbf{x}) - \langle g^2 \rangle), \quad (1)$$

with g gaussian. We show that the bound becomes very roughly $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 10^{-3}$ for the positive sign and $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 1$ for the negative sign.

It is known [1] that the form (1) can hold if the rise in $\mathcal{P}_\zeta(k)$ is generated after inflation by a curvaton-type mechanism. For the curvaton mechanism itself the sign is positive but it could be negative more generally. With a curvaton-type mechanism, k_{peak} corresponds to the horizon scale when the mechanism ceases to operate and black hole formation follows immediately.

A different possibility is for the rise to be generated by the waterfall field of hybrid inflation. A calculation has been done for a particular case [2], where simple expressions were found for ζ and \mathcal{P}_ζ . In that case ζ is gaussian and the black hole bound is well satisfied. The main purpose of this paper is to generalise the calculation of [2], to cover essentially any hybrid inflation model permitting exponential growth of the waterfall field.

II. BLACK HOLE BOUND

The bound that we are going to consider rests on the validity of the following statement: if, at any epoch after inflation, there are roughly spherical and horizon-sized regions with ζ significantly bigger than 1, a significant fraction of them will collapse to form roughly horizon-sized black holes.^{#1} The validity is suggested by the following argument: the overdensity at horizon entry is $\delta\rho/\rho \sim \zeta$, and if it is of order 1 then $\delta\rho \sim \rho = 3M_{\text{P}}^2 H^2$. The excess energy within the Hubble distance H^{-1} is then $M \sim H^{-3} \rho \sim M_{\text{P}}^2/H$, which means that the Hubble distance corresponds roughly to the Schwarzschild radius of a black hole with mass M . The validity is confirmed by detailed calculation using several different approaches, as summarised for instance in [3].

Before continuing we mention the following caveat. Practically all of the literature, as well as the simple argument just given, assumes that ζ within the region is not *very much* bigger than 1. Then the spatial geometry within the region is not too strongly distorted and the size of the black hole is indeed roughly that of the horizon. In the opposite case, the

^{#1} As in Eq. (1) we are choosing the background scale factor $a(t)$ so that the perturbation $\zeta = \delta(\ln a(\mathbf{x}, t))$ has zero spatial average.

background geometry is strongly distorted and the wavenumber k defined in the background no longer specifies the physical size of the region at the epoch $aH = k$ of horizon entry [4]. An entirely different discussion would then be necessary, which has not been given in the literature. As the opposite case does not arise in typical early-universe scenarios we ignore it.

We are interested in the case that $\mathcal{P}_\zeta(k)$ has a peak at some value k_{peak} , and we assume that the width of the peak in $\ln k$ is roughly of order 1 so that $\langle \zeta^2 \rangle \sim \mathcal{P}_\zeta(k_{\text{peak}})$. We will focus on the case that ζ is generated by the waterfall field perturbation. Then $\mathcal{P}_\zeta(k) \propto k^3$ below the peak with an exponential fall-off above the peak, giving^{#2}

$$\langle \zeta^2 \rangle \simeq \int_0^\infty \mathcal{P}_\zeta(k) dk/k = \frac{1}{3} \mathcal{P}_\zeta(k_{\text{peak}}), \quad (2)$$

but the numerical factor is not very important. What matters is that regions with $\zeta \gtrsim 1$ that might form black holes will be rare if $\mathcal{P}_\zeta(k_{\text{peak}})$ is sufficiently far below 1,

Observation demands that the regions must indeed be rare, because it places a strong upper bound on the fraction of β of space that can collapse to form black holes, on the assumption that the collapse takes place at a single epoch as is the case in our scenario. A recent investigation of the bound is given in [3], with extensive references to the literature. The bound can arise from many different physical effects, depending on the epoch of collapse, and is subject to many uncertainties. Typical bounds are roughly in the range

$$10^{-20} < \beta_{\text{max}} < 10^{-5}, \quad (3)$$

and we shall take that to be the range in what follows. To bound $\mathcal{P}_\zeta(k_{\text{peak}})$, we shall require $f < \beta_{\text{max}}$, where f is the fraction of space with $\zeta > 1$.

That fraction can be calculated from $\langle \zeta^2 \rangle$ if we know the probability distribution of $\zeta(\mathbf{x})$. The standard assumption is that it is gaussian. Then

$$f = \text{erfc}(1/\sqrt{2}\langle \zeta^2 \rangle^{1/2}), \quad (4)$$

and using the large- x approximation $\text{erfc}(x) \simeq e^{-x^2/2}$ we find $\langle \zeta^2 \rangle \simeq 1/2 \ln(1/f)$. For the range (3) this gives (with Eq. (2)) $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 0.01$ to 0.04 .

If instead ζ has the non-gaussian form (1) with the plus sign we have

$$\langle \zeta^2 \rangle = 2\langle g^2 \rangle^2 \simeq 2 \left[\frac{1}{2 \ln(1/f)} \right]^2, \quad (5)$$

which with Eq. (2) gives $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 6 \times 10^{-4}$ to 5×10^{-3} .

The situation when ζ has the form (1) with the minus sign is quite different. There is now no region of space where $\zeta > \langle g^2 \rangle$, and $f \ll 1$ now implies some bound $\langle g^2 \rangle - 1 \ll 1$ which is practically equivalent to $\langle g^2 \rangle < 1$. With Eq. (2) this corresponds to $\mathcal{P}_\zeta(k_{\text{peak}}) < 6$.

III. EVOLUTION OF THE WATERFALL FIELD

We adopt the notation and basic approach of [2]. In the first two subsections these are summarised, and then we present our new calculation of evolution of the waterfall field χ .

A. Standard hybrid inflation

We consider the usual hybrid inflation potential,

$$V(\phi, \chi) = V_0 + V(\phi) + \frac{1}{2} m^2(\phi) \chi^2 + \frac{1}{4} \lambda \chi^4 \quad (6)$$

$$m^2(\phi(t)) \equiv g^2 \phi^2(t) - m^2 \equiv g^2 (\phi^2(t) - \phi_c^2), \quad (7)$$

^{#2} If the integral fails to converge at small k we impose a cutoff corresponding to the size of the observable universe. The contribution from small k then has a negligible effect.

with $0 < \lambda \ll 1$ and $0 < g \ll 1$. We consider the standard case $m \gg H$. The effective mass-squared $m^2(\phi(t))$ of the waterfall field goes negative when the inflaton ϕ falls below $\phi_c \equiv m/g$. Then χ moves towards its vev, marking the beginning of what is called the waterfall. The waterfall is deemed to end when χ approaches its vev, which usually marks the end of inflation.

At least after the observable universe leaves the horizon, the inflaton potential $V(\phi(t))$ is supposed to have positive slope, and the inflaton is supposed to have zero vev so that $V(0) = 0$. Also, it is assumed that $V(\phi) \ll V_0$.

The requirements that V and $\partial V/\partial\chi$ vanish in the vacuum give the vev χ_0 and the inflation scale $V_0 \simeq 3M_{\text{P}}^2 H^2$:

$$\chi_0^2 = \frac{m^2}{\lambda} \simeq 12M_{\text{P}}^2 H^2 / m^2, \quad V_0 = \frac{m^4}{4\lambda} \simeq 3M_{\text{P}}^2 H^2. \quad (8)$$

The waterfall is supposed to begin with an era during which the evolution of χ is linear. Choosing a gauge whose slicing corresponds to uniform ϕ ,

$$\ddot{\chi}_{\mathbf{k}} + 3H\dot{\chi}_{\mathbf{k}} + [(k/a)^2 + m^2(\phi(t))] \chi_{\mathbf{k}} = 0. \quad (9)$$

The energy density and pressure of χ are

$$\rho_{\chi} = m^2(\phi)\chi^2 + \frac{1}{2}\dot{\chi}^2 + \frac{1}{2}|\nabla\chi|^2 \quad (10)$$

$$p_{\chi} = -m^2(\phi)\chi^2 + \frac{1}{2}\dot{\chi}^2 + \frac{1}{6}|\nabla\chi|^2. \quad (11)$$

In [2] we considered a regime of parameter space in which (i) the waterfall takes much less than a Hubble time, (ii) $m^2(t) \propto t$ during the linear era and (iii) χ is growing exponentially by the end of the linear era. We found simple formulas for the contribution ζ_{χ} of χ to the curvature perturbation at the end of the linear era, and for its spectrum $\mathcal{P}_{\zeta_{\chi}}$. In this paper we show that very similar formulas hold if we assume only exponential growth.

B. Exponential growth

Since inflation by definition continues during the waterfall, H will not vary much and we set it equal to a constant to simplify the presentation. Then conformal time is $\eta = -1/aH$ and (9) can be written

$$\frac{d^2(a\chi_{\mathbf{k}})}{d\eta^2} + \omega_k^2 a\chi_{\mathbf{k}} = 0, \quad (12)$$

with

$$\omega_k^2(\eta) \equiv k^2 + a^2\tilde{m}^2(t), \quad \tilde{m}^2 \equiv m^2(t) - 2H^2, \quad m^2(t) \equiv g^2\phi^2(t) - m^2. \quad (13)$$

For sufficiently small k , we can set $\omega_k^2 \simeq \omega_{k=0}^2 = a^2\tilde{m}^2$. Then ω_k^2 switches from positive to negative before $\phi = \phi_c$, but presumably not long before since $m \gg H$. For $k^2 > 0$ the switch is later. For the scales that we need to consider, we assume that there are eras both before and after the switch when ω_k^2 satisfies the adiabaticity condition $d|\omega_k^2|/d\eta \ll |\omega_k^2|$.

During the adiabaticity era before the switch we take the mode function to be

$$a\chi_k \simeq (2\omega_k(\eta))^{-1/2} \exp\left(-i \int_{\eta_1}^{\eta} \omega_k(\eta) d\eta\right), \quad (14)$$

which defines the vacuum state. During the adiabaticity era after the switch

$$a\chi_k \sim (2|\omega_k(\eta)|)^{-1/2} \exp\left(\int_{\eta_1(k)}^{\eta} |\omega_k(\eta)| d\eta\right), \quad (15)$$

where the subscript 1 denotes the beginning of the adiabatic era. The displayed prefactor holds [2] only if $m^2(t) \propto t$ and $H(t - t_1) \ll 1$ but its precise form doesn't matter. All we need is for it to vary sufficiently slowly that the growth is dominated by the exponential. We call the era during which that is true the growth era.

During the growth era, the adiabaticity condition is equivalent to the three conditions

$$\tilde{m}(t) \simeq m(t), \quad (16)$$

$$\frac{2H}{|m(t)|} \ll \left[1 - \left(\frac{k}{a(t)|m(t)|} \right)^2 \right]^{3/2} \quad (17)$$

$$\frac{1}{|m(t)|^2} \frac{d|m(t)|}{dt} \ll \left[1 - \left(\frac{k}{a(t)|m(t)|} \right)^2 \right]^{3/2}. \quad (18)$$

The adiabatic era begins when all three conditions are first satisfied.

By virtue of Eqs. (16) and (17), we have from Eq. (15) $\dot{\chi}_k \simeq |m(t)|\chi_k$. Except near places where $\chi(\mathbf{x}, t)$ vanishes, this implies

$$\dot{\chi}(\mathbf{x}, t) \simeq |m(t)|\chi(\mathbf{x}, t). \quad (19)$$

In the regime $k \ll a|m(t)|$ we have

$$|\omega_k| \simeq a|m(t)| \left(1 - \frac{1}{2} \frac{k^2}{a^2|m(t)|^2} \right) \quad (20)$$

giving

$$\chi_k(t) \simeq \chi_{k=0}(t) e^{-k^2/2k_*^2(t)}, \quad \chi_{k=0}(t) \simeq (2a^3|m(t)|)^{-1/2} \exp \left(\int_{t_1}^t dt |m(t)| \right), \quad (21)$$

where

$$k_*^2(t) \equiv \left(\int_{t_1}^t \frac{dt}{a^2|m(t)|} \right)^{-1}. \quad (22)$$

During the growth era, $\chi_k(t)$ is classical and proportional to the mode function $\chi_k(t)$. The same holds for $\chi(\mathbf{x}, t)$, except near places where it vanishes. The spectrum is $P_\chi = \chi_k^2$. Also,

$$\langle \chi^2(t) \rangle = \frac{4\pi}{(2\pi)^3} P_\chi(0, \tau) \int_0^\infty dk k^2 e^{-(k^2/k_*^2(t))} = (2\pi)^{-3/2} P_\chi(0, \tau) k_*^3(t), \quad (23)$$

and using the convolution theorem we have [2] for $k \ll k_*$ ^{#3}

$$\mathcal{P}_{\delta\chi^2}(t, k) = \frac{1}{\sqrt{\pi}} \langle \chi^2(t) \rangle^2 [k/k_*(\tau)]^3, \quad (24)$$

with $\mathcal{P}_{\delta\chi^2}(\tau, k)$ falling exponentially at $k \gg k_*$.

By virtue of Eq. (18), the change in $|m(t)|$ in time $|m(t)|^{-1}$ is negligible and so is the change in a . Setting $t = t_1 + |m(t)|^{-1}$ we get $k_*^2(t) \simeq a^2(t)|m(t)|^2$. But at this epoch the growth era has hardly begun, and subsequently $k_*(t)$ decreases while $a|m(t)|$ increases. So we really have $k_*^2(t) \ll a^2|m(t)|^2$ during the growth era, and χ_k falls exponentially in the regime $k_*(t) < k < a|m(t)|$ which means that $k_*(t)$ is the dominant mode.

We are mostly interested in an epoch just before the end of the linear era, which we denote by a subscript _{nl}. We denote $k_*(t_{\text{nl}})$ simply by k_* . Let us define $N_{\text{nl}} \equiv H(t_{\text{nl}} - t_1)$. If $N_{\text{nl}} \lesssim 1$,

$$1 \lesssim \frac{|m(t_1)|}{H} \lesssim \left(\frac{k_*}{a(t_{\text{nl}})H} \right)^2 \lesssim \frac{|m(t_{\text{nl}})|}{H} \leq \frac{m}{H} \quad (N_{\text{nl}} \lesssim 1). \quad (25)$$

^{#3} As usual $P_\chi \equiv (2\pi^2/k^3)\mathcal{P}_\chi$, with both P and \mathcal{P} referred to as the spectrum.

If instead $N_{\text{nl}} \gtrsim 1$, $k_*(t)$ levels out after $H(t - t_1) \sim 1$. We therefore have, whatever the value of N_{nl} ,

$$e^{-N_{\text{nl}}} \lesssim \left(\frac{k_*}{aH} \right)^2 \lesssim \frac{m}{H} e^{-N_{\text{nl}}}. \quad (26)$$

Since $|m(t)| \gg H$, the upper bound on k_* implies $(k_*/a)\chi_k \ll \dot{\chi}_k = |m(t)|\chi_k$. The spatial gradient of $\chi(\mathbf{x}, t)$ will therefore be small compared with the time-derivative, except near places where $\chi = 0$. Hence

$$\rho_\chi \simeq -\frac{1}{2}|m(t)|^2\chi^2 + \frac{1}{2}\dot{\chi}^2 \simeq 0 \quad (27)$$

$$p_\chi \simeq \frac{1}{2}|m(t)|^2\chi^2 + \frac{1}{2}\dot{\chi}^2 \simeq |m(t)|^2\chi^2 \simeq \dot{\chi}^2. \quad (28)$$

To evaluate ρ_χ we can use the local continuity equation (valid because the spatial gradient is negligible),

$$\dot{\rho}_\chi = -3H(\rho_\chi + p_\chi) \simeq -3Hp_\chi, \quad (29)$$

giving

$$\rho_\chi \simeq -(3H/2|m(t)|)p_\chi. \quad (30)$$

IV. CURVATURE PERTURBATION

Now we calculate the contribution of χ to the curvature perturbation, following closely the procedure of [2]. The curvature perturbation is given by $\zeta(\mathbf{x}, t) = \delta N(\mathbf{x}, t)$, where N is the number of e -folds of expansion from any initial slice with $a(\mathbf{x}, t) = a(t)$ (flat slice) to a slice of uniform ρ at time t . The contribution of χ during the linear era is $\zeta_\chi(\mathbf{x}) = \delta N(\mathbf{x}, t_{\text{nl}}, t_1)$, where $N(\mathbf{x}, t, t_1)$ is the expansion from a slice of uniform ρ just after the beginning of the linear era, to a slice of uniform ρ just before its end. We are working in a gauge where $\delta\phi = 0$ so that $\rho(\mathbf{x}, t) = \rho_\chi(\mathbf{x}, t) + \rho_\phi(t)$. Since $|\delta\rho_\chi(\mathbf{x}, t_1)| \ll |\delta\rho_\chi(\mathbf{x}, t_{\text{nl}})|$ we have

$$\zeta_\chi(\mathbf{x}, t) = -H \frac{\delta\rho_\chi(\mathbf{x}, t_{\text{nl}})}{\dot{\rho}(t)} = \frac{1}{3} \frac{\delta\rho_\chi(\mathbf{x}, t_{\text{nl}})}{\langle \dot{\chi}^2(t_{\text{nl}}) \rangle + \dot{\phi}^2(t)}, \quad (31)$$

where $\rho(t)$ is the spatial average of $\rho(\mathbf{x}, t)$.

Using the equations of the previous section this gives for $k \ll k_*$

$$\zeta_\chi(\mathbf{x}) = -\frac{H}{2|m(t_{\text{nl}})|} \frac{\langle \dot{\chi}^2(t_{\text{nl}}) \rangle}{\langle \dot{\chi}^2(t_{\text{nl}}) \rangle + \dot{\phi}^2(t_{\text{nl}})} \frac{\delta\chi^2(\mathbf{x}, t_{\text{nl}})}{\langle \chi^2(t_{\text{nl}}) \rangle}, \quad (32)$$

and

$$\mathcal{P}_{\zeta_\chi}(k) \simeq \left[\frac{H}{2|m(t_{\text{nl}})|} \frac{\langle \dot{\chi}^2(t_{\text{nl}}) \rangle}{\langle \dot{\chi}^2(t_{\text{nl}}) \rangle + \dot{\phi}^2(t_{\text{nl}})} \right]^2 \left(\frac{k}{k_*} \right)^3. \quad (33)$$

At $k \gg k_*$, \mathcal{P}_{ζ_χ} is negligible because χ_k is. The spectrum therefore peaks at $k \sim k_*$.

If $\langle \chi^2(t_{\text{nl}}) \rangle \gg \dot{\phi}^2(t_{\text{nl}})$ we have $\dot{\rho} \simeq \dot{\rho}_\chi$, which means that the slice of uniform ρ is practically the same as the one of uniform ρ_χ . In turn, that is the same as the slice of uniform χ .

In general the formula $\zeta = \delta N$ holds only after smoothing ρ and p on a scale big enough that the local continuity equation is satisfied. In our case though, that equation is satisfied on all of the scales $k \gg k_*$ on which ζ_χ is significant. Therefore, the formula $\zeta_\chi(\mathbf{x}) = \delta N(\mathbf{x}, t_{\text{nl}}, t_1)$ makes sense on all of these scales. In the opposite regime $k \ll k_*$, ζ_χ will be negligible simply because $\delta\chi^2$ is, its exponential growth not having begun. Therefore, \mathcal{P}_{ζ_χ} peaks at k_* whether that scale is super-horizon or sub-horizon.

We want to see whether the black hole bound of Section II is satisfied by Eq. (33). The situation is simple if k_* is super-horizon. Then, ζ is of the form Eq. (1) with the minus sign, and the black hole bound is $\mathcal{P}_{\zeta_\chi}(k_*) < 6$ which is well satisfied.

If instead k_* is sub-horizon, we have to remember that the black hole bound refers to horizon-sized regions. To apply it, we must drop sub-horizon modes of ζ_χ . Estimating the bispectrum, trispectrum as in [2], one sees that this makes ζ_χ nearly gaussian. The black hole bound is therefore roughly $\mathcal{P}_{\zeta_\chi}(H) \lesssim 10^{-2}$. Since $k_* \gg k_{\text{end}}$ and $|m(t)| \gg H$, it too is presumably satisfied.

Earlier calculations of ζ_χ are reviewed in [2], including the one in [5]. A continuation of [5] has since appeared [6]. These papers consider the potential $V(\phi) \propto \phi^2$, for which $m^2(t) \propto t$ is a good approximation. Their expression for ζ_χ (Eq. A(10) of [5] and Eq. (6.15) of [6]) is the same as our Eq. (32), except that the middle factor is missing. However, the meaning of $\delta\chi$ and $\langle\chi^2\rangle$ in the final factor is different in their expression from that in Eq. (32); χ in their case is smoothed on the horizon scale whereas in our case it is not.

V. CONCLUSION

We have calculated the contribution to ζ generated during the linear era of the waterfall, assuming that such an era exists and that exponential growth takes place during that era. Such growth can occur only if the tachyonic mass of the waterfall field is $m \gg H$ (standard hybrid inflation) which we therefore demand.

The calculation generalises [2], to the case of an arbitrary time-dependence of $m^2(t)$ subject to it being slow enough to allow exponential growth. Because the dependence is arbitrary, we cannot repeat the detailed investigation given in [2], of the parameter space within which our calculation will apply. That could only be done with a specific inflaton potential $V(\phi)$, which among other things would determine $m^2(t)$.

We have, as usual, considered only super-horizon modes of the curvature perturbation ζ . The usual super-horizon treatment of ζ , based on the local energy continuity equation does still hold for sub-horizon modes during the waterfall. But we have not used these modes, and to do so would require evolution of the cosmological perturbations through to the post-inflation era. In particular, that would be necessary if we were to discuss the possible formation of black holes whose size is much smaller than the horizon scale at the end of inflation.

VI. ACKNOWLEDGMENTS

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